THE IRREDUCIBILITY OF CERTAIN PURE-CYCLE HURWITZ SPACES

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Abstract. We study "pure-cycle" Hurwitz spaces, parametrizing covers of the projective line having only one ramified point over each branch point. We start with the case of genus-0 covers, using a combination of limit linear series theory and group theory to show that these spaces are always irreducible. In the case of four branch points, we also compute the associated Hurwitz numbers. Finally, we give a conditional result in the higher-genus case, requiring at least $3g$ simply branched points. These results have equivalent formulations in group theory, and in this setting complement results of Conway-Fried-Parker-Völklein.

1. Introduction

In this paper, we use a combination of geometric and group-theoretic techniques to prove a result with equivalent statements in both fields. The geometric statement is that certain genus-0 Hurwitz spaces (the "pure-cycle" cases) are always irreducible, while the group-theoretic statement is that the corresponding factorizations into cycles always lie in a single pure braid group orbit. "Pure-cycle" refers to the hypothesis that for our covers, there is only a single ramified point over each branch point. The main significance for us of this condition is that it allows us to pass relatively freely between the point of view of branched covers, where one moves the branch points freely on the base curve, and linear series, where one moves the ramification points freely on the covering curve. This facilitates induction, as it is easier to stay within the pure-cycle case from the point of view of linear series.

Our result is close to optimal in the sense that if one drops either of the pure-cycle or genus-0 hypotheses, one quickly runs into cases where the Hurwitz spaces have more than one component. However, we do prove a conditional generalization to higher-genus pure-cycle Hurwitz spaces having at least $3g$ simply branched points, depending on a positive answer to a different geometric question which is closely related to an old question of Zariski.

Our immediate motivation for studying the pure-cycle situation is its relation to linear series: specifically, if one wishes to prove statements on branched covers via linear series arguments, the pure-cycle situation is the natural context to examine. A good understanding of the classical situation is therefore important to studying other cases, such as that of positive characteristic. In particular, our main theorem allows for a much simpler proof of a stronger result in [13] than would otherwise be possible. However, we also remark that a good understanding of the components of Hurwitz spaces has given rise to a wide range of substantial applications: the

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classical proof of Severi that $\mathcal{M}_g$ is connected [11]; number-theoretic applications such as Davenport's problem, and Thompson's genus-0 problems in group theory, both due to Fried [7]; and the Fried-Völklein description of the absolute Galois group of certain fields in inverse Galois theory [9],[10]. Furthermore, our results in particular should provide good test cases for Fried's Main Conjecture of modular tower theory [8]. See also the survey article [5].

We now state our results more precisely. We will recall/fix our terminology in the next section.

The following proposition is well known, although the equivalence of the first two and last two conditions depends heavily on the fact that we restrict our attention to covers with a single ramified point over each branch point. We will recall the proof in the following section.

Proposition 1.1. *Given d and $\vec{e} = (e_1, \dots, e_r)$ with $2d - 2 = \sum_i (e_i - 1)$, the following are equivalent:*

a) *the Hurwitz factorizations for $(d, r, 0, \vec{e})$ all lie in a single orbit of the pure braid group.*
b) *the space $\mathcal{H}(d, r, 0, \vec{e})$ is irreducible, where $\mathcal{H}(d, r, 0, \vec{e})$ is the Hurwitz space parametrizing r distinct points $Q_1, \dots, Q_r$ on $\mathbb{P}^1$ together with a genus-0 cover of $\mathbb{P}^1$, such that each Q_i has a single point over it ramified to order e_i, and the rest unramified;*
c) *the space $MR := MR(\mathbb{P}^1, \mathbb{P}^1, \vec{e})$ is irreducible, where MR is the space parametrizing r distinct points $P_1, \dots, P_r$ on $\mathbb{P}^1$ together with a rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d and ramified to order e_i at P_i (on the source curve) for all i;*
d) *the space $G^1_d := G^1_d(\mathbb{P}^1, \vec{e})$ is irreducible, where G^1_d is the space parametrizing r distinct points $P_1, \dots, P_r$ on $\mathbb{P}^1$ together with a linear series of dimension 1 and degree d, having ramification e_i at P_i for all i;*

Our main theorem is then the following:

Theorem 1.2. *Given d, r and $e_1, \dots, e_r$ with $2d - 2 = \sum_i (e_i - 1)$, the equivalent conditions of Proposition 1.1 always hold.*

Our proof follows the general structure of Eisenbud and Harris' argument in [3], where they prove the irreducibility of certain families of linear series without prescribed ramification. However, while they work exclusively from the perspective of linear series, we have to switch back and forth between points of view. Starting from the perspective of linear series, we use a degeneration argument and the tools of limit linear series to reduce to a base case of four points on $\mathbb{P}^1$, and then solve that case directly, after switching to the group-theoretic point of view of Hurwitz factorizations. Our explicit work in the base case also computes the Hurwitz numbers for that case:

Theorem 1.3. *Given d and $\vec{e} = (e_1, \dots, e_4)$ with $2d - 2 = \sum_i (e_i - 1)$, we have the following formula for the Hurwitz number:*

$$h(d, r, 0, \vec{e}) = \min\{e_i(d + 1 - e_i)\}_i.$$

We can also describe the Hurwitz factorizations in this case completely explicitly.

Lastly, in Theorem 5.5 below, we again use limit linear series techniques to prove a conditional version of Theorem 1.2 for pure-cycle cases of higher genus having at

least $3g$ simply branched points, depending on a positive answer to Question 5.4 below, a geometric question closely related to an old question of Zariski.

The higher-genus result could be seen as having the spirit of an effective version in the pure-cycle case of results of Conway-Fried-Parker-Völklein. Our main theorem also generalizes a theorem of Fried [4, Thm. 1.2], which implies the case of our Theorem 1.2 in which $e_i = 3$ for all i.

Finally, we remark that the combination of the genus-0 and pure-cycle conditions imply that our monodromy groups are always either cyclic, S_d, or A_d; we show this, independently of the proof of our main results, in Theorem 5.3 below.

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2. Notation and terminology

We quickly recall terminology and fix our notation. For geometric statements, we assume throughout that we are working over $\mathbb{C}$.

Our notation for permutations will always be to express them as products of cycles. Given $\sigma \in S_d$, we will say that a number $k \in \{1, \dots, d\}$ is **in the support** of σ if $\sigma(k) \neq k$.

Given a permutation σ (or conjugacy class T) of S_d, we define its **index** $\iota(\sigma)$ as follows: if $a_1 \leq a_2 \leq \cdots \leq a_m$ is the corresponding partition, then $\iota(\sigma) := \sum_{i=1}^{m}(a_i - 1)$. We then say that a tuple $(d, r, g, (T_1, \dots, T_r))$ constitutes the data of a **Hurwitz problem**, where $d \geq 1, r \geq 2, g \geq 0$, the T_i are conjugacy classes in S_d, and we require $2d - 2 + 2g = \sum_i \iota(T_i)$.

Associated to a Hurwitz problem we have the group-theoretic question of finding all **Hurwitz factorizations** $(\sigma_1, \dots, \sigma_r)$, where:

(i) $\sigma_i \in T_i$;
(ii) $\sigma_1 \dots \sigma_r = 1$;
(iii) the σ_i generate a transitive subgroup of S_d.

We say that two Hurwitz factorizations are **equivalent** if they are related by simultaneous conjugation by an element of S_d. We call the number of equivalence classes of Hurwitz factorizations the **Hurwitz number** $h(d, r, g, (T_1, \dots, T_r))$. See Remark 2.2 below for discussion of some relating and conflicting notation in the literature.

Geometrically, we also have the **Hurwitz space** $\mathcal{H}(d, r, g, (T_1, \dots, T_r))$, parametrizing r-tuples of marked points on $\mathbb{P}^1$, together with covers of degree d and genus g, unramified away from the marked points, and with monodromy type T_i at the ith marked point for all i. For a fixed choice of marked points, such covers correspond to Hurwitz factorizations up to equivalence, so the degree of $\mathcal{H}(d, r, g, (T_1, \dots, T_r))$ over the space $\mathcal{M}_{0,r}$ parametrizing marked points is given by the Hurwitz number.

We say that a Hurwitz problem is **pure-cycle** if each T_i consists of a single cycle. Throughout this paper, we restrict our attention to pure-cycle Hurwitz problems, and we replace the T_i by integers $e_i \geq 2$ giving the length of the cycle. We thus have $2d - 2 + 2g = \sum_i (e_i - 1)$ as the condition on our data.

We recall that the **Artin braid group** B_r acts on tuples $(\sigma_1, \dots, \sigma_r)$ in S_d with $\sigma_1 \dots \sigma_r = 1$, preserving the group generated by the σ_i. The ith generator acts by

replacing (σ_i, σ_{i+1}) by $(\sigma_{i+1}, \sigma_{i+1}^{-1}\sigma_i\sigma_{i+1})$. The kernel of the natural map $B_r \to S_r$ is the **pure braid group**, which not only preserves $\sigma_1 \dots \sigma_r = 1$, but sends each σ_i to a conjugate of itself in the group generated by all the σ_i. We thus see that the pure braid group acts on the set of Hurwitz factorizations, and it is the orbits of this action which we will study.

Note as a consequence of the geometric definition of Hurwitz number that the number is clearly invariant under reordering of the e_i. We can also see this purely in terms of group theory by making use of the braid group action to permute the e_i arbitrarily.

We will also be working from the point of view of **linear series**, which from our point of view will always have dimension 1 and be basepoint free: in this situation, a linear series of dimension 1 and degree d (also called a $\mathfrak{g}^1_d$) on a curve C is simply a map to $\mathbb{P}^1$ of degree d, considered up to automorphism of the image space. We remark that the basepoint-free hypothesis will not cause us any problems, as we will always be working with spaces of linear series with all ramification specified.

As a simple case of the sort of analysis we will carry out in the four-point case, we recall the answer in the case of three points:

Lemma 2.1. *The Hurwitz number for* $(d, 3, 0, (e_1, e_2, e_3))$ *is always* 1*, corresponding to the factorization:*

$$\begin{aligned}
&\sigma_1 = (d - e_2, d - e_2 - 1, \dots, 2, 1, e_3, e_3 + 1, e_3 + 2, \dots, d - 1, d),\\
&\sigma_2 = (d, d - 1, \dots, d - e_2 + 2, d - e_2 + 1), \text{ and}\\
&\sigma_3 = (1, 2, \dots, e_3 - 1, e_3).
\end{aligned}$$

Proof. First, note that by transitivity and the fact that $\sigma_1 = \sigma_3^{-1}\sigma_2^{-1}$, we have that σ_3 and σ_2 together act non-trivially on all of $\{1, \dots, d\}$, and their actions must therefore overlap on a subset of cardinality exactly $e_2 + e_3 - d = d + 1 - e_1$.

To complete the proof, one observes that if a sequence of precisely m consecutive elements in the cycle representation of σ_2 also appear in σ_3, at most $m - 1$ of them can remain fixed by $\sigma_2\sigma_3$. It follows that in order for $\sigma_2\sigma_3$ to be an e_1-cycle, the overlap must form a single contiguous portion of each of σ_2 and σ_3, from which one easily concludes the desired statement. □

Finally, we recall:

Proof of Proposition 1.1. The equivalence of (i) and (ii) is classical and quite general: the basic idea is that the monodromy cycles of a cover depend not only on the cover, but also on a choice of local monodromy generators of the fundamental group of the base; all such choices of generators are related by braid operations, and each braid operation can be achieved as monodromy of the Hurwitz space by moving the marked points of the base around one another. For a slightly different exposition, see [18, Prop. 10.14 (a)]; note that the situation is slightly different because he considers Hurwitz spaces with unordered branch points and full braid orbits, but the argument is the same in our case of ordered branch points and pure braid orbits.

Similarly, the equivalence of (iii) and (iv) is equally basic: the space G^1_d is obtained from the space MR simply by modding out by the (free) action of the automorphism group of the base $\mathbb{P}^1$, so MR is a PGL_2-bundle over G^1_d, and one space is irreducible if and only if the other is.

Next, because we have restricted to Hurwitz spaces in which there is a single ramified point over each branch point, the comparison of MR and $\mathcal{H}(d, r, 0, \vec{e})$ is almost equally straightforward. First suppose $r \geq 3$. If we denote by $\widehat{MR}$ the open subscheme of MR for which the map f sends the marked ramification points to distinct points, then because $r \geq 3$, we have that $\widehat{MR}$ is a PGL_2-bundle over $\mathcal{H}(d, r, 0, \vec{e})$, so one is irreducible if and only if the other is. But then an easy deformation-theory argument shows that any component of MR dominates the $(\mathbb{P}^1)^r$ parametrizing the branch points of the map f [15, Cor. 3.2], so we see that $\widehat{MR}$ is dense in MR, completing the desired equivalences for irreducibility. Finally, if $r = 2$, the only maps are, up to automorphism, $x \mapsto x^d$, so it is easy to see that both MR and $\mathcal{H}(d, r, 0, \vec{e})$ are irreducible. □

Remark 2.2. Our terminology of Hurwitz problem (and more specifically, the associated set of Hurwitz factorizations) is closely related to the more standard terminology "Nielsen class", for which one also specifies a subgroup G which the σ_i must generate, and assigns the T_i as conjugacy classes within that subgroup.

The Nielsen class is frequently better because it gives a finer combinatorial invariant: the Hurwitz factorizations for a given Hurwitz problem are a disjoint union over different Nielsen classes, and likewise the Hurwitz space is a disjoint union over spaces associated to different Nielsen classes. Our main theorem immediately implies that for the cases we study, a Hurwitz problem consists of only a single Neilsen class. See also Theorem 5.3 below for a direct proof of this fact.

One has to be slightly careful in comparing statements, since the Nielsen class terminology also allows for different equivalence relations on the Hurwitz factorizations (for instance, working up to inner automorphism of G).

We also remark that our terminology of Hurwitz number, although standard in some areas, conflicts with the usage in [6]. Specifically, in *loc. cit.*, the term "Hurwitz number" is used to describe the number of components of the Hurwitz space, while what we call the Hurwitz number is called the "degree".

3. Reduction to four points

The goal of this section is to use the machinery of limit linear series to prove:

Proposition 3.1. *To prove Theorem 1.2 in general, it is enough to give a proof in the case that* $r = 4$.

In order to use a degeneration argument for Proposition 3.1, the key fact which we need (and which is lacking in the higher-genus case) is:

Proposition 3.2. *Every component of the space* G^1_d *of Proposition 1.1 maps dominantly under the forgetful map to* $\mathcal{M}_{0,r}$.

Proof. Indeed, we know [1, Thm. 2.3] that if we fix ramification points, we have only finitely many $\mathfrak{g}^1_d$'s with the prescribed ramification, and that conversely, if we move the branch points, our rational function can always be deformed [15, Cor. 3.2]; the statement then follows by a dimension count, as in the proof of *ibid.* □

We will make essential use of the $r = 1$ case of limit linear series, developed by Eisenbud and Harris in [2]. We briefly review the critical points of their theory in this case, where it becomes considerably simpler. Suppose that we have a family $\mathcal{C}$ of curves, with smooth generic fiber, but with some nodal fibers. We assume

that all nodal fibers are of compact type, i.e., that their dual graph is a tree. Eisenbud and Harris construct a space over all of $\mathcal{C}$ which correspond to usual $\mathfrak{g}^1_d$'s on smooth fibers of $\mathcal{C}$, but correspond to *limit linear series* on the nodal fibers; by abuse of notation, we write $\mathfrak{g}^1_d$ to mean also limit linear series. Suppose that C is a nodal fiber with (necessarily smooth) components $C_1, \dots, C_m$. In our case of $r = 1$, a (refined) limit linear series on C may be expressed as an m-tuple of *aspects* on each C_i, where an aspect is a $\mathfrak{g}^1_{d_i}$ with $d_i \leq d$, and the sole compatibility condition is that if C_i and C_j meet at a node P, then the ramification index at P of the aspects on C_i and C_j should be the same. Given r smooth sections P_i of $\mathcal{C}$, the Eisenbud-Harris construction also works to give spaces of $\mathfrak{g}^1_d$'s with at least a specified amount of ramification at the P_i (in fact, limit linear series should in general allow for base points away from the nodes, but since we will work with the case that all ramification is specified, this won't arise).

We review the situation further in the case $g = 0$, with all ramification specified. This is studied in [14, Thm. 2.4]; there, the families considered involve only breaking off one component at a time, but our assertions here easily follow by the same arguments. For the rest of the section, we fix our degenerate curve:

Situation 3.3. The curve C_0 is the totally degenerate curve given by a nodal chain of $r-2$ copies of $\mathbb{P}^1$, with P_1, P_2 on the first component, P_i on the $(i-1)$st component for $i < 2 < r-1$, and P_{r-1}, P_r on the last component.

We consider families $\mathcal{C}$ near a fiber isomorphic to the specified C_0. Because all ramification is specified, the space of $\mathfrak{g}^1_d$'s is finite over $\mathcal{C}$, and is in fact finite étale in a neighborhood of C_0. Furthermore, a $\mathfrak{g}^1_d$ on C_0 is uniquely described by a collection of ramification indices $(e'_2, \dots, e'_{r-2})$ at the nodes, which are required to satisfy a collection of triangle inequalities and a parity condition. Specifically, if we consider any consecutive triple e, e', e'' starting with an odd-indexed term in the sequence

$$e_1, e_2, e'_2, e_3, \dots, e_{r-2}, e'_{r-2}, e_{r-1}, e_r,$$

we need to have $e \leq e' + e''$, $e' \leq e + e''$, and $e'' \leq e + e'$, and we need $e + e' + e''$ to be odd.

For later use, we note that the second condition implies immediately that the triangle inequalities are in fact always strict, and also that the allowed parity of $e'_2, \dots, e'_{r-2}$ is fixed by the e_i.

With these tools in hand, we can now complete our geometric argument.

Proof of Proposition 3.1. We fix the totally degenerate curve C_0 as in the above situation, and work with a local universal family $\mathcal{C}$ of genus-0 curves in a neighborhood of C_0, denoting the generic curve of this family (which is also the generic curve of $\mathcal{M}_{0,r}$) by C_η. It is enough to show that the relative G^1_d space (with the desired ramification at the marked points) is irreducible over the family $\mathcal{C}$, since by the previous proposition, every component of the global G^1_d space meets the generic curve C_η. By the same token, it is enough to show that any two $\mathfrak{g}^1_d$'s on the geometric generic fiber $\bar{C}_\eta$ lie on the same component of G^1_d. Furthermore, because the space of $\mathfrak{g}^1_d$'s is reduced over C_0, we have cannot have two components of G^1_d meet over C_0, so over our family $\mathcal{C}$, irreducible components of G^1_d are the same as connected components.

Accordingly, suppose we are given two $\mathfrak{g}^1_d$'s on $\bar{C}_\eta$. By the above discussion, these can be specialized to $\mathfrak{g}^1_d$'s on C_0, which are described by the data of ramification

indices $(e'_2, \ldots, e'_{r-2})$ and $(e''_2, \ldots, e''_{r-2})$ respectively. We set the convention that $e'_1 = e''_1 := e_1$, and $e'_{r-1} = e''_{r-1} := e_{r-1}$. Our claim is as follows: if we assume the $r = 4$ case of Theorem 1.2, then any two $\mathfrak{g}^1_d$'s on C_0 such that $e'_i = e''_i$ for all but one i necessarily lie on the same component of G^1_d.

Indeed, if we fix a node of C_0 corresponding to e'_i (i.e., the $(i-1)$st node), we can restrict the family $\mathcal{C}$ to the closed subfamily $\mathcal{C}_i$ in which only the chosen node of C_0 is allowed to be smoothed, giving a smooth component containing the two marked points P_i and P_{i+1}, and the $(i-2)$nd and ith nodes (unless $i = 2$ or $r-2$, in which case P_1 or P_r takes the place of the $(i-2)$nd or ith node respectively). The other components remain fixed, so we may consider $\mathcal{C}_i$ to be obtained from the universal family over $\overline{\mathcal{M}}_{0,4}$ by localizing around a degenerate curve, and gluing appropriate chains of $\mathbb{P}^1$'s at the first and fourth marked points; in particular, the base of $\mathcal{C}_i$ is naturally a local scheme U of $\overline{\mathcal{M}}_{0,4}$ at a point corresponding to a degenerate curve. If we write $\mathcal{C}_{0,4}$ for the universal curve over U, the point is to relate the G^1_d spaces associated to $\mathcal{C}_{0,4}$ and $\mathcal{C}_i$.

Specifically, suppose we have chosen indices $e'_j = e''_j$ for all $j \neq i$. For the sake of clarity, we denote by $G^1_d(\mathcal{C})$ our original space of $\mathfrak{g}^1_d$'s on $\mathcal{C}$, and by $G^1_d(\mathcal{C}_i)$ and $G^1_d(\mathcal{C}_{0,4})$ the spaces of $\mathfrak{g}^1_d$'s on $\mathcal{C}_i$ and $\mathcal{C}_{0,4}$. For the first two spaces, we impose ramification e_i at each P_i, so that $G^1_d(\mathcal{C}_i)$ is simply the base change of $G^1_d(\mathcal{C})$, while for $G^1_d(\mathcal{C}_{0,4})$ we impose ramification $e'_{i-1}, e_i, e_{i+1}, e'_{i+1}$ at the four marked points. Now, if we consider the closed subscheme Z_i of $G^1_d(\mathcal{C}_i)$ which corresponds to limit $\mathfrak{g}^1_d$'s with ramification indices e'_j at the nodes (for $j \neq i$), the limit $\mathfrak{g}^1_d$'s are uniquely determined except on the component with four marked points, so Z_i is isomorphic to the space $G^1_d(\mathcal{C}_{0,4})$ which we have described. Thus if we assume Theorem 1.2 in the case $r = 4$, we see that the subscheme Z_i of $G^1_d(\mathcal{C}_i)$ is irreducible, so that any two $\mathfrak{g}^1_d$'s on C_0 for which $e'_i = e''_i$ for all but one i lie on the same connected component of $G^1_d(\mathcal{C}_i)$, and hence of $G^1_d(\mathcal{C})$.

This proves the claim, and since every limit $\mathfrak{g}^1_d$ on C_0 can be smoothed to a $\mathfrak{g}^1_d$ on $\bar{C}_\eta$, the following numerical lemma completes the proof of our proposition. $\square$

Lemma 3.4. *Let C_0 be a totally degenerate marked curve of genus 0, and suppose we are given two $\mathfrak{g}^1_d$'s with ramification indices e_i at the marked points, and classified by ramification indices $(e'_2, \ldots, e'_{r-2})$ and $(e''_2, \ldots, e''_{r-2})$ respectively at the nodes. Then it is possible to modify $(e'_2, \ldots, e'_{r-2})$ into $(e''_2, \ldots, e''_{r-2})$, by a sequence of changes affecting only one index at a time, and with every intermediate set of indices corresponding to a valid $\mathfrak{g}^1_d$ on C_0.*

Proof. Suppose we have a $\mathfrak{g}^1_d$ on C_0 specified by the set $(e'_2, \ldots, e'_{r-2})$. Since the allowed parity of each of $e'_2, \ldots, e'_{r-2}$ is fixed by the e_i, as long as we change them by 2 at a time, we do not need to worry about violating the parity condition. It is thus enough to show that if $(e'_2, \ldots, e'_{r-2})$ and $(e''_2, \ldots, e''_{r-2})$ are distinct, there is always some i with $e'_i \neq e''_i$ and for which we can increase e'_i or e''_i to make it closer to the other without violating any triangle inequalities. We prove this by induction.

We will induct on the following statement: suppose we are given i such that $e''_i - e'_i \geq e''_{i-1} - e'_{i-1}$ and $e'_i + 2 \leq e'_{i-1} + e_i$. Then either we can increase e'_i, or we must have $e''_{i+1} - e'_{i+1} \geq e''_i - e'_i$ and $e'_{i+1} + 2 \leq e'_i + e_{i+1}$. Indeed, if we cannot increase e'_i, the only triangle inequalities that could be violated are $e'_i + 2 \leq e'_{i-1} + e_i$ or $e'_i + 2 \leq e_{i+1} + e'_{i+1}$. But the first one is satisfied by hypothesis, so the only possibility

is that $e'_i + 2 > e_{i+1} + e'_{i+1}$, in which case we see we must have $e'_i + 1 = e_{i+1} + e'_{i+1}$. But we then see that

$$e''_{i+1} - e'_{i+1} = e''_{i+1} + e_{i+1} - e'_i - 1 \geq e''_i - e'_i$$

by the triangle inequality. Furthermore, $e'_{i+1} + 2 \leq e'_i + e_{i+1}$ because $e_{i+1} \geq 2$.

Now suppose that i_0 is the smallest number with $e'_{i_0} \neq e''_{i_0}$. Without loss of generality, we may assume that $e'_{i_0} < e''_{i_0}$. But we see that this satisfies the hypotheses of our inductive statement: the first inequality is clear since $e''_{i_0-1} = e'_{i_0-1}$, while the second follows because we have $e'_{i_0} + 2 \leq e''_{i_0} \leq e''_{i_0-1} + e_{i_0} = e'_{i_0-1} + e_{i_0}$. But by induction, we see that we must eventually be able to increment one of the e'_i for $i \geq i_0$, since when $i = r-2$, we have $e''_{r-1} = e'_{r-1} = e_{r-1}$. This proves the lemma. □

4. The case of four points

In this section, we study the case of four points from the group-theoretic point of view. Our setup throughout this section is as follows:

Situation 4.1. We are given $d > 0$, and $\vec{e} := (e_1, e_2, e_3, e_4)$, with $2d-2 = \sum_i (e_i - 1)$, and $2 \leq e_1 \leq e_2 \leq e_3 \leq e_4 \leq d$.

We observe for later use that in our situation, we have $e_1 + e_3 \leq d+1$, $e_2 + e_4 \geq d+1$, $e_1 + e_2 \leq d+1$, and $e_3 + e_4 \geq d+1$. The first two inequalities follow from $e_1 + e_3 \leq e_2 + e_4$ together with $e_1 + e_2 + e_3 + e_4 = 2d+2$, while the second two follow by comparing with the first two.

Throughout this section, we will write sequences of the form $i, i+1, \ldots, j$ (and similarly for descending sequences). If $j \geq i$, the meaning is clear: an ascending sequence of length $j-i+1$. However, without further comment we will also allow $j = i-1$, in which case the meaning will be the empty sequence (still of length $j-i+1$).

Our main result is the following:

Theorem 4.2. *In Situation 4.1, the Hurwitz number $h(d, 4, 0, \vec{e})$ is given by $\min\{e_i(d+1-e_i)\}_i$.*

Moreover, the possible Hurwitz factorizations $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ are classified explicitly as follows:

(i) *if $\sigma_3\sigma_4$ is trivial or a single cycle, then we have*

$$\sigma_1 = (d, d-1, \ldots, e_3 + e_4 + 1 - k,$$
$$\sigma^{-(d+2-k-e_1)}(\ell), \sigma^{-(d+3-k-e_1)}(\ell), \ldots, \sigma^{-(e_3+e_4+1-2k)}(\ell) = \ell),$$
$$\sigma_2 = (e_3 + e_4 + 1 - k, e_3 + e_4 + 2 - k, \ldots, d-1, d, \ell, \sigma^{-1}(\ell), \ldots, \sigma^{-(d+1-k-e_1)}(\ell)),$$
$$\sigma_3 = (k, k-1, \ldots, 2, 1, e_4 + 1, e_4 + 2, \ldots, e_3 + e_4 - k),$$
$$\sigma_4 = (1, \ldots, e_4),$$

where we allow any k with $e_3 + e_4 - d \leq k \leq e_3$ and $k \leq d+1-e_2$, we define $\sigma := (k, k+1, \ldots, e_3 + e_4 - k) = \sigma_3\sigma_4$, and for a given k, we allow ℓ to vary in the range $k \leq \ell \leq e_3 + e_4 - k$.

(ii) *if $\sigma_3\sigma_4$ is a product of two disjoint cycles, then we have*

$$\sigma_1 = (m+e_1-1, m+e_1-2, \dots, m+1, m),$$
$$\sigma_2 = (d, d-1, \dots, m+e_1, m+d+k-e_3-e_4, m+d-1+k-e_3-e_4, \dots, k),$$
$$\sigma_3 = (k, k-1, \dots, 1, e_4+1, e_4+2, \dots, m+e_1-1,$$
$$m, m-1, \dots, m+d+1+k-e_3-e_4, m+e_1, m+e_1+1, \dots, d),$$
$$\sigma_4 = (1, \dots, e_4),$$

where we allow any k with $1 \leq k \leq e_3+e_4-d-1$, and any m with $e_4-e_1+1 \leq m \leq d+1-e_1$ and $m \leq e_4$.

Before giving the proof, we give a number of simple technical lemmas and their consequences; although each result individually is quite easy and presumably well-known, we include them for the sake of staying as self-contained as possible.

We begin by simplifying the transitivity condition on Hurwitz factorizations in our situation.

Lemma 4.3. *Suppose that $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in S_d$ are cycles of length e_1, e_2, e_3, e_4, with trivial product. Then the following are equivalent:*

a) *the σ_i form a Hurwitz factorization for $(d, 4, 0, \vec{e})$;*
b) *every number in $\{1, \dots, d\}$ is in the support of at least one of the σ_i;*
c) *every number in $\{1, \dots, d\}$ is in the support of exactly two of the σ_i, except that either there is some k which is in the support of every σ_i, or there exist $k \neq \ell$, with each in the support of three of the σ_i.*

Proof. It is clear that the transitivity condition for a) implies b). To see that b) implies c), the key point is that the identity $2d-2 = \sum_i (e_i - 1)$ leaves relatively little flexibility for the σ_i. Specifically, if every number in $\{1, \dots, d\}$ is in the support of at least one cycle, it must be in the support of at least two, as otherwise the product could not be trivial. But we have $\sum_i e_i = 2d+2$, and $2d$ of the numbers in the support of the σ_i are accounted for, leaving only 2 which could be in the support of more than two cycles. We conclude c).

Finally, to see that c) implies a), we need only check transitivity. Since every number is in the support of at least two σ_i, we cannot have any σ_i disjoint from all the others. Thus, the only way they could fail to generate a transitive subgroup would be if two of the σ_i were disjoint from the other two. But this cannot occur, as the inequalities $e_3 + e_4 \geq d+1$ and $e_2 + e_4 \geq d+1$ imply that σ_4 cannot be disjoint from either of σ_2 or σ_3. □

We next pursue a detailed study of the relationship between pairs of cycles and their products.

Lemma 4.4. *Suppose $\sigma_1, \sigma_2 \in S_d$ are non-disjoint cycles in S_d, and let σ be any cycle in the decomposition of $\sigma_1\sigma_2$ into disjoint cycles. Then there exists an element of $\{1, \dots, d\}$ in the support of σ, σ_1, and σ_2.*

Proof. This is routine: if σ consisted entirely of numbers in the support of σ_1 but not σ_2, it would have to be equal to σ_1, contradicting the non-disjointness hypothesis, and similarly with the σ_i reversed. One then verifies that to switch from elements in the support of σ_1 to elements in the support of σ_2 requires an element of σ in the support of both. □

Lemma 4.5. *Let σ, σ' be non-disjoint cycles, with $\sigma\sigma' \neq 1$. Then there exists a unique expression (up to cycling of indices) of σ' as $(w'_1, v'_1, \dots, w'_m, v'_m)$ and σ as $(w_1, v_1, w_2, v_2, \dots, w_m, v_m)$ where the w'_i, v'_i and w_i, v_i are sequences of numbers, satisfying:*

(i) *the w_i and w'_i are all non-empty, but the v_i and v'_i may be empty;*
(ii) *each v'_i consists of numbers not in the support of σ;*
(iii) *each v_i consists of numbers not in the support of σ';*
(iv) *there exists a permutation $\tau \in S_m$ such that each w_i is the inverse of $w'_{\tau(i)}$ (i.e., the same sequence in reversed order);*
(v) *if for all i we set k_i to be the first number in w_i, the set of k_i is precisely the set of numbers in the support of all three of σ, σ', and $\sigma\sigma'$.*

Proof. By Lemma 4.4, there is some number in the support of σ, of σ' and of $\sigma\sigma'$; we begin by designating one such number to be k_1. In order to be able to write σ in the desired form, the order of the remaining k_i are then uniquely determined. Furthermore, we see that each w_i must consist of the longest word in σ which starts with k_i, contains only numbers also in the support of σ', and does not contain k_j for $j \neq i$. This uniquely determines each w_i, and the v_i are what remain. We can then do the same for the w'_i and v'_i, except that the k_i could appear in a different order in σ', giving us the permutation τ. It remains to check that these expressions have the desired properties, specifically (ii), (iii), and (iv).

Note that if $n \neq k_i$ for any i is any number in the support of σ and σ', since n isn't in the support of $\sigma\sigma'$, then $\sigma'(n) = \sigma^{-1}(n)$, so we see that $\sigma'(n)$ must also be in the support of σ, immediately prior to n in the cycle representation. Applying this inductively gives that all such n appear in the w_i in σ and in the w'_i in σ', and that each $w'_{\tau(i)}$ is inverse to w_i, as desired. □

The following corollary is quite special to the case of at most two repetitions.

Corollary 4.6. *Let σ, σ' be cycles, and write $S \subseteq \{1, \dots, d\}$ for the intersection of the supports of σ,σ' and $\sigma\sigma'$. Suppose that either:*

(I) *$\sigma = \sigma_3$, $\sigma' = \sigma_4$, $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ a Hurwitz factorization for $(d, 4, 0, \vec{e})$;*
(II) *σ and σ' are not disjoint, and $\#S \leq 2$.*

Then the number of disjoint cycles in $\sigma\sigma'$ is equal to $\#S$ and is at most 2, and there is exactly one element of S in the support of each disjoint cycle.

Proof. We first observe that (I) implies (II). Indeed, σ_3 and σ_4 cannot be disjoint since $e_3 + e_4 \geq d + 1$, and we have $\#S \leq 2$ by Lemma 4.3 c), since any k which occurs in the support of $\sigma_3\sigma_4$ must also occur in the support of σ_1 or σ_2 because of the trivial-product condition.

We next argue that (II) implies the conclusion of the corollary. Lemma 4.4 immediately handles the case $\#S \leq 1$. For $\#S = 2$, we apply the above lemma, noting first that in this case we can always cycle indices so that $\tau = 1$, i.e., each w'_i is the inverse of w_i. Then note that the formula

$$(w_1, v_1, w_2, v_2)(w'_1, v'_1, w'_2, v'_2) = (k_1, v'_1, v_2)(k_2, v'_2, v_1)$$

holds regardless of whether the v_i and v'_i have non-zero length; since k_1 and k_2 are assumed to be in the support of $\sigma\sigma'$, we see that it must consist of two disjoint cycles. □

We are now ready to give the proof of Theorem 4.2. For the sake of clarity, we break the proof into four propositions.

Proposition 4.7. *Each of the possibilities enumerated in Theorem 4.2 gives a valid Hurwitz factorization, and $\sigma_3\sigma_4$ is in the asserted form (and in particular, consists of the asserted number of cycles).*

Proof. The main technicality is to check that the individual cycles themselves make sense. This involves checking three points: first, that all the elements listed lie in $\{1, \dots, d\}$; second, that each word has non-negative length; and third, that there is no overlap between the words in a given cycle. In fact, we first see that all words have non-negative length, which then simplifies checking that their ranges are appropriate.

Before that, we observe that $\sigma = (k, k+1, \dots, e_3 + e_4 - k)$ makes sense: the length is at least 1, since we have $2k \leq 2e_3 \leq e_3 + e_4$; and we check both $k \geq 1$ and $e_3 + e_4 - k \leq d$ using the inequality $e_3 + e_4 - d \leq k$ (together with $e_3 + e_4 \geq d + 1$ for the first).

In general, we allow sequences to have length 0, except those containing k, ℓ, or m. In case (i), for σ_1 we require that $d \geq e_3+e_4-k$ and $e_3+e_4+1-2k \geq d+2-k-e_1$, which are equivalent to $k \geq e_3 + e_4 - d$ and $k \leq d + 1 - e_2$ respectively. For σ_2, we need $e_3 + e_4 - k \leq d$ and $d + 1 - k - e_1 \geq 0$, giving $k \geq e_3 + e_4 - d$ and $k \leq d + 1 - e_1$ respectively. Since $d + 1 - e_2 \leq d + 1 - e_1$, the last inequality will also be satisfied. Finally, for σ_3 we need $k \geq 1$ and $e_3 + e_4 - k \geq e_4$; the first is satisfied since $e_3 + e_4 - d \geq 1$, while the second is simply $k \leq e_3$. Case (ii) is similar, with the only inequality appearing other than those imposed directly being $m \geq e_3 + e_4 - d$. However, this is okay, since we have $m \geq e_4 - e_1 + 1$, and the inequality $e_1 + e_3 \leq d + 1$ implies that $e_3 + e_4 - d \leq e_4 - e_1 + 1$. Thus, the ranges provided guarantee that the cycles make sense, and are in fact equivalent to having the lengths of all words containing k, ℓ, or m be at least 1, and the lengths of the remaining words being at least 0.

We next address the first and third points simultaneously. In case (i), everything follows easily from the ranges imposed for k: for σ_4 there is nothing to check; for σ_3 we check that $k < e_4 + 1$ and $e_3 + e_4 - k \leq d$; and for σ_2 and σ_1, everything is immediate, since the terms involving σ are automatically in the correct range, and the sequence in σ_1 involving σ could not wrap around without the sequence in σ_2 having negative length, and vice versa.

Similarly, in case (ii), the only points requiring any non-immediate checking are: for σ_3, that $k < m + d + 1 + k - e_3 - e_4$, and $m < e_4 + 1$, with the former following from $e_1 + e_3 \leq d + 1$; for σ_2, that $m + d + k - e_3 - e_4 < m + e_1$; and for σ_1, that $m \geq 1$. Thus, all the cycles consist of non-overlapping entries in $\{1, \dots, d\}$.

We can then check directly that the cycles are of the correct length and have trivial product, as well as that $\sigma = \sigma_3\sigma_4$. Finally, using that b) implies a) in Lemma 4.3 makes it easy to check that the cycles generate transitive subgroups of S_d, so all the possibilities listed are valid Hurwitz factorizations.

It remains only to note that in case (i), we have already written σ explicitly, so we see that $\sigma_3\sigma_4$ is in fact trivial or a single cycle, while in case (ii), we check that $m > m + d + k - e_3 - e_4$, so that σ_1 is disjoint from σ_2, and since $\sigma_1\sigma_2\sigma_3\sigma_4 = 1$, it follows that $\sigma_3\sigma_4$ is a product of two disjoint cycles. $\square$

Proposition 4.8. *No two possibilities enumerated in Theorem 4.2 are equivalent.*

Proof. Cases (i) and (ii) of Theorem 4.2 are clearly invariant under relabeling. In case (i), we see that k is determined as the number of elements in the support of both σ_3 and σ_4, so is invariant under relabeling. If $\sigma = 1$ (i.e., if $k = e_3+e_4-k$), we have $\ell = k$ is the only possibility. Given k with $\sigma \neq 1$, we see that ℓ is determined as the unique number (in the allowed range) such that $\sigma^{\ell-k}(k)$ is in the support of σ_1 and σ_2 (and necessarily σ), so two possibilities with different ℓ cannot be equivalent.

In case (ii), the size of the intersection of the supports of σ_1 and σ_4 is e_4+1-m, so m is relabeling-invariant. The overlap between the supports of σ_3 and σ_4 consists of two contiguous words, and k is determined as the length of the word with non-empty overlap with σ_2. Hence, no two possibilities are equivalent. □

Proposition 4.9. *Every Hurwitz factorization is equivalent to one of the possibilities enumerated in Theorem 4.2.*

Proof. We begin by noting that by Corollary 4.6, we must have that $\sigma_3\sigma_4$ consists of 0, 1, or 2 disjoint cycles. Furthermore, if $\sigma_3\sigma_4 = 1$, then we have $\sigma_3 = \sigma_4^{-1}$, and $\sigma_1 = \sigma_2^{-1}$, and $e_1 = e_2 = d+1-e_3 = d+1-e_4$, so it is easy to check that the only possibility is the $k = \ell = e_3 = e_4$ case of (i). We can thus assume that $\sigma_3\sigma_4 \neq 1$.

The first case we consider is that $\sigma = \sigma_3\sigma_4$ is a single cycle, or, equivalently by Corollary 4.6, that there is a single number $k' \in \{1, \dots, d\}$ which is in the support of σ_3, σ_4, and in $\sigma = \sigma_3\sigma_4$. Let k be the number of elements in the support of both σ_3 and σ_4. We may then relabel so that $\sigma_4 = (1, \dots, e_4)$, and k' gets mapped to k; i.e, so that the unique number in the support of σ_3, σ_4, and σ is k. Applying Lemma 4.5 to σ_3 and σ_4 with the only k_i being k gives us that σ_3 is necessarily of the form $(k, k-1, \dots, 2, 1, a_1, \dots, a_{e_3-k})$ for some $a_i \in \{e_4+1, \dots, d\}$; relabeling the latter range allows us to put σ_3 in the desired form.

Next, note that by Lemma 4.3, there must be a unique number ℓ in the support of σ_1, of σ_2, and of σ. We then have also by Lemma 4.3 that all the numbers $\{e_3+e_4-k+1, \dots, d\}$ must be in the support of σ_2, and we claim that they must be in a contiguous word, and followed immediately by $\ell, \sigma^{-1}(\ell), \dots, \sigma^{-(d+1-k-e_1)}(\ell)$. The claim is checked by applying Lemma 4.5 to σ_2 and σ, using that $\sigma_2\sigma = \sigma_1^{-1}$, so that the only k_i is $k_1 = \ell$. The claim implies that we are free to reorder $\{e_3 + e_4 - k + 1, \dots, d\}$ so that they appear in order, and furthermore so that $\sigma_2(d) = \ell$. Hence, we have put σ_2 in the desired form, and then σ_1 is determined by $\sigma_1\sigma_2\sigma = 1$.

We next consider the case that σ is a product of two disjoint cycles, which by Corollary 4.6 is equivalent to having two numbers $k', k'' \in \{1, \dots, d\}$ which are each in the support of σ_3, σ_4, and in $\sigma := \sigma_3\sigma_4$. Then k' is in one of the disjoint cycles of σ, and k'' is in the other. By Lemma 4.3, we see that since we already have k', k'' occurring in σ_3, σ_4 and σ (hence in either σ_1 or σ_2), we cannot have any numbers occurring in σ_1, σ_2 and σ. By Corollary 4.6 (II), we see that σ_1 and σ_2 must be disjoint, and since $\sigma_1\sigma_2 = \sigma^{-1}$, we see that k' is in the support of one, and k'' is in the support of the other; without loss of generality, we may assume that k' is in the support of σ_1 and k'' in σ_2. We also note that this implies that each of $\{1, \dots, d\}$ is in the support of either σ_3 or σ_4.

We once again normalize so that $\sigma_4 = (1, \dots, e_4)$, and we can further require that if we write $\sigma_3 = (w_1, v_1, w_2, v_2)$ and $\sigma_4 = (w_1', v_1', w_2', v_2')$ as in Lemma 4.5, we can set $w_1' = (1, 2, \dots, k)$, with k being the corresponding relabeling of k'', i.e., the

unique number in the support of σ_3,σ_4, and σ_2. We then have w_1 in the desired form, and w_2 will likewise be in the desired form for some m, which will necessarily be the unique number in the support of σ_3,σ_4, and σ_1. Relabelling $e_4+1,\dots,d$ as necessary, we can place v_1 and v_2, hence σ_3 in the desired form, and σ_1 and σ_2 are then uniquely determined as disjoint cycles with $\sigma_1\sigma_2\sigma = 1$, and containing m and k respectively.

This then completes the proof of the claim that every Hurwitz factorization is equivalent to one of the enumerated possibilities. □

Proposition 4.10. *The number of possibilities enumerated in Theorem 4.2 is equal to* $\min\{e_i(d+1-e_i)\}_i$.

Proof. The formula $\min\{e_i(d+1-e_i)\}_i$ falls into two situations: if $e_4 \geq d+1-e_1$, then it is equal to $e_4(d+1-e_4)$, while if $e_4 \leq d+1-e_1$, then it gives $e_1(d+1-e_1)$.

We first consider the situation that $e_4 \geq d+1-e_1$. Here, because $e_4+e_1 \geq d+1$, we have $e_2+e_3 \leq d+1$, so $e_3 \leq d+1-e_2$, and in case (i) of Theorem 4.2 the inequality $e_3+e_4-d \leq k \leq e_3$ automatically implies $k \leq d+1-e_2$. We thus have

$$\sum_{k=e_3+e_4-d}^{e_3} \sum_{\ell=k}^{e_3+e_4-k} 1 = \sum_{k=e_3+e_4-d}^{e_3} (e_3+e_4-2k+1) = (d+1-e_3)(d+1-e_4)$$

possibilities from case (i). Similarly, we have $d+1-e_1 \leq e_4$ so $e_4-e_1+1 \leq m \leq d+1-e_1$ implies that $m \leq e_4$. Thus, our ranges are $1 \leq k \leq e_3+e_4-d-1$ and $e_4-e_1+1 \leq m \leq d+1-e_1$, yielding $(e_3+e_4-d-1)(d+1-e_4)$ possibilities in case (ii), and giving us the desired $e_4(d+1-e_4)$ possibilities in total (note that e_3+e_4-d-1 and $d+1-e_4$ are always non-negative, so these formulas are always valid).

The situation that $e_4 \leq d+1-e_1$ proceeds similarly, with e_1e_2 possibilities arising from case (i), and $e_1(d+1-e_1-e_2)$ possibilities arising from case (ii). □

Combining the statements of the four propositions, we immediately conclude Theorem 4.2.

From the theorem, we deduce quite directly:

Corollary 4.11. *In Situation 4.1, the Hurwitz factorizations for* $(d,4,0,\vec{e})$ *all lie in a single pure braid orbit.*

Proof. We first see that all the factorizations in case (i) of the theorem are in a single pure braid orbit, and then show that any factorization in case (ii) is in the same braid orbit as some factorization in case (i).

Suppose we start with $(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ corresponding to a given k,ℓ of case (i). Our first claim is that if we replace (σ_1,σ_2) by $(\sigma_2^{-1}\sigma_1\sigma_2, \sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_2 = \sigma\sigma_2\sigma^{-1})$, we stay in case (i), leaving k fixed, while replacing ℓ by $\sigma(\ell)$. The first part is clear, while the assertion on ℓ is checked by direct computation, using that since σ_3,σ_4 remain fixed, it is enough to see what happens to σ_2. Thus, for a given k, every possible ℓ is in the same braid orbit.

To analyze the Hurwitz factorizations for different k, for each k we set $\ell = k$, where we have $\sigma_1 = (d, d-1, \dots, e_3+e_4+1-k, d+1-e_2, d-e_2, \dots, k)$ and hence

$$\sigma_2\sigma_3 = \sigma_1^{-1}\sigma_4^{-1} = (k, k-1, \dots, 1, e_4, e_4-1, \dots, d+2-e_2,$$
$$e_3+e_4+1-k, e_3+e_4+2-k, \dots, d).$$

We check that if we replace (σ_2, σ_3) by $(\sigma_2', \sigma_3') := (\sigma_3^{-1}\sigma_2\sigma_3, \sigma_3^{-1}\sigma_2^{-1}\sigma_3\sigma_2\sigma_3)$, then as long as k is not minimal, we remain in case (i), but replace k by $k-1$. Here, a relabeling is in principle necessary, but we can instead check that $\sigma_3'\sigma_4$ is still a single cycle, so that we remain in case (i), and that the supports of σ_3' and σ_4 overlap in $k-1$ elements. We therefore see that every possibility in (i) is always in a single pure braid orbit.

Finally, we suppose we have $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ corresponding to a given k, m of case (ii). In this case, we again replace (σ_2, σ_3) by $(\sigma_2' = \sigma_3^{-1}\sigma_2\sigma_3, \sigma_3' = \sigma_3^{-1}\sigma_2^{-1}\sigma_3\sigma_2\sigma_3)$, and note that since σ_1, σ_4 remain unchanged, σ_2' determines σ_3'. One then computes that as long as $k < e_3 + e_4 - d - 1$, σ_2' is still a possibility from case (ii), with m the same, but $k+1$ instead of k. Finally, if $k = e_3 + e_4 - d - 1$, one checks that applying the same pure braid operation, we move into case (i), with $k = e_3 + e_4 + d$ (and $\ell = m$). Thus, every possibility in case (ii) is in the same pure braid orbit as some possibility in case (i), and we get that everything is in the same pure braid orbit. □

Using Proposition 1.1, and Proposition 3.1, we see immediately that Corollary 4.11 implies Theorem 1.2, and we are done.

5. Loose ends

We begin with a further remark in the case of four points. The Hurwitz number $\min\{e_i(d+1-e_i)\}_i$ computes the number of rational functions $\mathbb{P}^1 \to \mathbb{P}^1$ with four fixed branch points on the target. If instead we look at fixed ramification points on the source, we find that the number is $\min\{e_i, d+1-e_i\}_i$ [14, Rem. 5.9]. Despite the close geometric relationship between these two numbers, there is no *a priori* reason for there to be any numerical relationship at all, so their similarity is striking. We note further that with the exception of the case that we have $e_i = d$ for some d, both formulas are symmetric with respect to replacing the e_i by $d+1-e_i$; this motivates us to ask:

Question 5.1. Is there a natural involution on the set of rational functions of degree d having exactly four ramification points, which replaces the ramification indices e_i by $d+1-e_i$, and holds both the ramification and branch points fixed?

A more obvious question left unanswered by our analysis is:

Question 5.2. Is there a closed form for the genus-0 pure-cycle Hurwitz numbers for any number of branch points?

Next, we observe that it is a consequence of Theorem 1.2 that if we fix d, r and $\vec{e}$, all possible Hurwitz factorizations are in a single Nielsen class, i.e., they generate the same group, and are in the same conjugacy classes within that group. However, with some non-trivial group theory and sufficient perseverance, one can already see this quite directly:

Theorem 5.3. *Given d, r and $\vec{e} = (e_1, \dots, e_r)$ with $2d-2 = \sum_i (e_i - 1)$, and all $e_i \geq 2$, suppose we have $(\sigma_1, \dots, \sigma_r)$ and $(\sigma_1', \dots, \sigma_r')$ two Hurwitz factorizations for $(d, r, 0, \vec{e})$, generating groups $G, G' \subseteq S_d$. Then there exists a simultaneous conjugation in S_d making $G' = G$, and each σ_i conjugate to σ_i' inside G. That is, any two Hurwitz factorizations lie in the same Nielsen class.*

In fact, if $r = 2$, we have G isomorphic to the cyclic group C_d. If $r = 3$ with $(e_1, e_2, e_3) = (4, 4, 5)$, we have $G \cong S_5$, imbedded as a doubly transitive subgroup of

S_6. Otherwise, we always have $G = S_d$ or $G = A_d$ depending on the parity of the e_i.

Proof. The case that $r = 2$ is clear, as we must have $e_1 = e_2 = d$.

For $r = 3$, we note that the first assertion is clear, since the Hurwitz number is equal to 1 by Lemma 2.1.

For $r > 3$, we reduce the first assertion to the second. In the case that $G = S_d$, this is trivial, while in the case that $G = A_d$, we need only observe that since $2d - 2 = \sum_i (e_i - 1)$, and all $e_i \geq 3$, we can have at most one cycle of order greater than $d - 2$. We can always fix this cycle by simultaneous conjugation in S_d, and then any cycles of given length less than or equal to $d - 2$ are in the same conjugacy class in A_d.

For the second assertion, we begin by arguing that with $r > 2$, we must have G primitive, i.e., that there is no non-trivial partition of $\{1, \ldots, d\}$ into blocks on which the action of G is well-defined. Indeed, if there were such a partition, since G is transitive the blocks would all have to have the same size m, for some $m|d$. We would then necessarily have each σ_i either of size a multiple of m, acting as a $e'_i := \frac{e_i}{m}$-cycle σ'_i on $d' := \frac{d}{m}$ blocks of size m, or of size strictly less than m, acting trivially on the blocks. Say we have s of the latter; without loss of generality, we may assume that $e_1, \ldots, e_s < m$, and $e_{s+1}, \ldots, e_r \geq m$. Then $\sigma'_{s+1}, \ldots, \sigma'_r$ give a Hurwitz factorization in $S_{d'}$, so we must have $2d' - 2 \leq \sum_{i=s+1}^{r} (e'_i - 1)$. On the other hand, we compute that since $2d - 2 = \sum_i (e_i - 1)$, we have $2d + r - 2 = \sum_i e_i$, so

$$2\frac{d}{m} + \frac{r-2}{m} - \sum_{i=1}^{s} \frac{e_i}{m} = \sum_{i=s+1}^{r} \frac{e_i}{m} = \sum_{i=s+1}^{r} e'_i,$$

and so $2d' - 2 - \delta = \sum_{i=s+1}^{r} (e'_i - 1)$, where

$$\delta = \sum_{i=1}^{s} \frac{e_i}{m} - s + r - 2 - \frac{r-2}{m} \geq \frac{2s}{m} - s + r - 2 - \frac{r-2}{m} = \frac{(m-1)(r-s-2)}{m} + \frac{s}{m},$$

so we must have $\frac{(m-1)(r-s-2)}{m} + \frac{s}{m} \leq 0$. Since the σ'_i act transitively on d' elements, and have trivial product, there must be at least 2 of them which are non-trivial, so that $r - s - 2 \geq 0$. Since $m > 1$, we see that $\delta \geq 0$, and we can have $\delta = 0$ only if $r - s - 2 = s = 0$, i.e., $r = 2$. Thus, with our hypothesis that $r > 2$, we must have $\delta > 0$, a contradiction.

We note that in the case that $d \leq 3$, the only transitive subgroups are A_d and S_d, so there is nothing to prove. In the case $d = 4$, one checks directly that there is no primitive subgroup other than S_4 and A_4, so we need only consider the case $d \geq 5$.

Now, we wish to apply the theorem of Williamson [19] stating that if a primitive subgroup of S_d contains a cycle of order e, with $e \leq (d - e)!$, then it must be either A_d or S_d. Since we have $2d - 2 = \sum_i (e_i - 1)$, we see that we must have $e_i \leq \lfloor \frac{2d-2}{r} + 1 \rfloor$ for some i. One then computes directly that Williamson's theorem gives the desired result unless we have $r = 3, d \leq 10$, or $r = 4, d \leq 5$. More specifically, the only cases falling outside Williamson's theorem are $r = 3$ with $(e_1, e_2, e_3) = (3, 4, 4), (4, 4, 5), (5, 5, 5), (7, 7, 7)$ or $r = 4$ with $(e_1, e_2, e_3, e_4) = (3, 3, 3, 3)$. In these cases, one can check directly that the group is A_d or S_d, as appropriate, except in the $(4, 4, 5)$ case, where one can compute the group explicitly, checking that it is doubly transitive and has order 120, which is well-known to determine it uniquely. $\square$

Our result is sharp in the sense that if one drops either the pure-cycle or the genus-0 hypothesis, there are many examples for which the Hurwitz space is not irreducible. However, there are nonetheless many examples for which the Hurwitz space is irreducible which are not covered by our main theorem. We will consider here one generalization which remains in the pure-cycle case, but seeks to drop the genus-0 hypothesis in favor of an assumption that could be viewed philosophically as an effective form of the results of Conway-Fried-Parker-Völklein, in that it requires at least $3g$ transpositions in order to apply. However, our result will be conditional on a positive answer to a geometric question, which we now discuss.

Zariski asked whether every Hurwitz space of genus-g covers of $\mathbb{P}^1$ with prescribed branching type over at least $3g$ points maps dominantly to $\mathcal{M}_g$ under the forgetful map. This is now known to be false in some cases, but we will be interested in an analogous yet different question which arises when one wants to compare the points of view of linear series and branched covers:

Question 5.4. Fix $r, g \geq 0$, $d \geq 1$ and $\vec{e} = (e_1, \dots, e_r)$ with $2 \leq e_i \leq d$ for all i, and $2d - 2 - g = \sum_i (e_i - 1)$. Consider the space MR parametrizing tuples consisting of a genus-g curve C, points $P_1, \dots, P_{r+3g}$ on C, and a map $f : C \to \mathbb{P}^1$ of degree d, ramified to order e_i at P_i for $i \leq r$ and simply ramified at $P_{r+1}, \dots, P_{r+3g}$. Does every component of MR map dominantly to $\mathcal{M}_{g,r}$ under the map induced by forgetting f and $P_{r+1}, \dots, P_{r+3g}$?

The positive answer to this question in the case $g = 0$ is Proposition 3.2. We also remark that Steffen [17] (see also [12]) has a result along these lines for linear series of any degree and dimension, but without any ramification specified. He accomplishes this by studying degeneracy loci of suitable maps of vector bundles; one might try to study our question by looking at Schubert conditions on maps of vector bundles, and suitable intersections of such conditions.

The application of Question 5.4 to irreducibility of Hurwitz spaces is as follows.

Theorem 5.5. *Fix r, g, d, and $\vec{e}$ as above. Then a positive answer to Question 5.4 implies that $\mathcal{H}(d, r, g, \vec{e})$ is irreducible, where $\mathcal{H}(d, r, g, \vec{e})$ is the Hurwitz space of covers of $\mathbb{P}^1$ of genus g and degree d, with a single ramified point of index e_i over the ith branch point for $i \leq r$, and simple branching over the remaining branch points. Equivalently, the set of Hurwitz factorizations consisting of e_i-cycles and $3g$ transpositions all lie in a single pure braid orbit.*

Proof. We first consider the generalization of Proposition 1.1 in this case. The argument for the equivalence of (i) and (ii) goes through unmodified in the generality of higher-genus covers. The argument for the equivalence of (iii) and (iv), where in both cases we prescribe simple ramification at $3g$ additional unspecified points, is likewise the same as in the genus 0 case. We then have that the Hurwitz space is the image of (a dense open subset of) MR, so we see that (iii) or (iv) imply (i) and (ii), and it is enough to check (iv), i.e., to work from the point of view of linear series.

A positive answer to Question 5.4 takes the place of Proposition 3.2, and allows us to work over the generic r-marked curve of genus g, or more specifically, locally around a given degenerate curve, as in the genus 0 case. Instead of working with a totally degenerate curve, we work with a curve C_0 consisting of a copy of $\mathbb{P}^1$ with r marked points, and with g elliptic tails. As in the proof of [14, Thm. 2.6], the limit linear series on this curve are completely determined by their aspects on $\mathbb{P}^1$;

on each elliptic tail, they consist of the degree 2 map to $\mathbb{P}^1$, simply ramified at the node (and at three other points, which are uniquely determined as differing from the node by 2-torsion points). Furthermore, the ramification imposed at each node on $\mathbb{P}^1$ is simple ramification; thus, the limit linear series are in natural bijection with the linear series on $\mathbb{P}^1$ with the prescribed ramification at $r+g$ points. We know by Theorem 1.2 that the space of these linear series is irreducible as we allow the $r+g$ ramification points to move, so we conclude irreducibility of the space of $\mathfrak{g}^1_d$'s in a neighborhood of C_0, and in particular, on the generic r-marked curve of genus g, as desired. $\square$

Results of Conway-Fried-Parker-Völklein (see [9, Appendix], and also [5]) show that, roughly speaking, for any given group and collection of conjugacy classes, if every conjugacy class is repeated often enough, then the components of the Hurwitz space are determined by a certain invariant, called the lifting invariant. Our results fit into the same general philosophy, and might be thought of as an effective version of Conway-Fried-Parker-Völklein for the pure-cycle case.

Finally, we remark that Question 5.4 would potentially have interesting applications to the study of covers in positive characteristic, as well. One cannot hope for a positive answer outside characteristic 0 without some further hypotheses: for instance, in the case $g=0$, the statement is known to fail if one does not require all $e_i < p$ (see [16, Ex. 5.6]). However, a positive answer in the case all $e_i < p$ would give an important step towards giving new non-existence results for tame covers in positive characteristic, as is carried out in the genus-0 case in [13].

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